Universality in 2D CFT and 3D Gravity

How to Toast your Bagels Perfectly Every Time

David Grabovsky

UCSB – Berenstein Group Meeting

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Outline

The Partition Function in QFT

Consider an arbitrary QFT on $S^1_L \times \{\text{time}\}$ at temperature $T = \frac{1}{\beta}$ with Hamiltonian H. The Hilbert space $\mathcal{H}$ consists of states on S^1_L, so H has a discrete spectrum $\{E_n\}$ bounded below. The partition function is

$$Z_L(\beta) \equiv \mathrm{Tr}_{\mathcal{H}}\left(e^{-\beta H}\right) = \sum_n \langle n|e^{-\beta H}|n\rangle = \sum_n e^{-\beta E_n}. \qquad (1.1)$$

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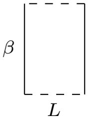
$$Z_L(\beta) = \int_{T^2(L,\beta)} \mathcal{D}\phi\, e^{-S[\phi]} =$$



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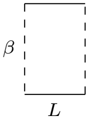

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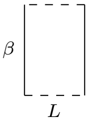

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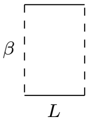

$$= \int_{T^2(\beta,L)} \mathcal{D}\phi \, e^{-S[\phi]} = Z_\beta(L). \tag{1.2}$$

This relation is completely general. It relates two *different* theories with the same Euclidean action but on spatial manifolds of different sizes.

The Partition Function in CFT

If our theory is scale-invariant, then $Z_L(\beta) = L_{aL}(a\beta)$, and only the "bagel aspect ratio" $\frac{\beta}{L}$ matters. Thus we set $L = 2\pi$. It follows that

$$Z(\beta) \equiv Z_{2\pi}(\beta) = Z_\beta(2\pi) = Z_{2\pi}\left(2\pi \cdot \frac{2\pi}{\beta}\right) = Z\left(\frac{4\pi^2}{\beta}\right). \tag{1.3}$$

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The UV spectrum and thermodynamics are determined by the IR.
But the IR is determined largely by the ground state, which is *universal*.

More generally, β is replaced by $\tau = \frac{i\beta}{2\pi} \in \mathbb{C}$, and in fact $Z(\tau) = Z(-\frac{1}{\tau})$. Even more generally, $Z(\tau)$ is invariant under $\mathrm{PSL}(2,\mathbb{Z})$ transformations.

Operators and their Dimensions

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- The central charge is $c = c_L + c_R$, and we assume that $c_L = c_R$.
- The dimension of $\mathcal{O}$ is $\Delta = h + \bar{h}$, while its spin is $J = h - \bar{h}$. We will treat only the case $J = 0$ of zero angular potential.
- In radial quantization, the Hamiltonian is a dilation operator, $H = D - \frac{c}{12}$, shifted by the Casimir energy $E_0 = -\frac{c}{12}\frac{2\pi}{L}$.
- The eigenvalues of H are E; the eigenvalues of D are Δ.
- The ground state $|0\rangle$ corresponds to the identity operator $\mathbb{1}$, which is the unique operator with $\Delta_0 = 0 \iff E_0 = -\frac{c}{12}$.

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Including degeneracies, $Z(\beta) = \sum_E \rho(E) e^{-\beta E} = \sum_\Delta \rho(\Delta) e^{-\beta\left(\Delta - \frac{c}{12}\right)}$,

where the density of states $\rho(E) = e^{S(E)}$ is related to the entropy.

Outline

The Thermal Partition Function

Goal: to study the thermodynamics of 2D CFTs at high temperature.

High temperature: $\beta \longrightarrow 0$. Then $Z(\beta) \longrightarrow \sum_E \rho(E) e^{-0E} \sim N \sim e^S$.

Low temperature: $\beta \longrightarrow \infty$. Then $Z(\beta) \longrightarrow \sum_E e^{-\infty E} = e^{-\beta E_0} = e^{\frac{\beta c}{12}}$.

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Low temperature: $\beta \longrightarrow \infty$. Then $Z(\beta) \longrightarrow \sum_E e^{-\infty E} = e^{-\beta E_0} = e^{\frac{\beta c}{12}}$.

These two regimes are linked by modular invariance. At high temperature,

$$Z(\beta) = Z\left(\frac{4\pi^2}{\beta}\right) = e^{\frac{c}{12}\cdot\frac{4\pi^2}{\beta}} = e^{\frac{\pi^2 c}{3\beta}} \implies \boxed{\log Z(\beta) = \frac{\pi^2 c}{3\beta}.} \qquad (2.1)$$

This statement is *asymptotic*: it's a good enough approximation at high enough temperatures, but is silent on *when* it's valid and *how good* it is.

Thermodynamics and Spectrum

Let's make like undergraduates and compute! The free energy $F(\beta)$ is

$$\log Z(\beta) = \frac{\pi^2 c}{3\beta} = \beta F(\beta) \implies F(\beta) = \frac{\pi^2 c}{3\beta^2} = \frac{c}{12}\left(\frac{2\pi}{\beta}\right)^2. \qquad (2.2)$$

The thermodynamic entropy $S(\beta)$ at high temperature is obtained from

$$S(\beta) = \left(1 - \beta\partial_\beta\right)\log Z(\beta) = \frac{2\pi^2 c}{3\beta}. \qquad (2.3)$$

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We change to the microcanonical ensemble via $\langle E\rangle_\beta = -\partial_\beta \log Z(\beta)$:

$$\boxed{S(E) = 2\pi\sqrt{\frac{c}{3}E} \iff \rho(\Delta) = \exp\left[2\pi\sqrt{\frac{c}{3}\left(\Delta - \frac{c}{12}\right)}\right]}. \tag{2.4}$$

This is the **Cardy formula**. It says that at high energies, **all unitary, modular invariant 2D CFTs have universal thermodynamics.**

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$$\log Z(\beta) = \frac{c}{12}\begin{cases} \frac{4\pi^2}{\beta}, & \beta \ll 2\pi, \\ \beta, & \beta \gg 2\pi. \end{cases} \tag{2.5}$$

These results are only valid away from the self-dual point $\beta_* = 2\pi$, but they might remind you of **(!)** the Hawking–Page transition.

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3. We've just solved the **bootstrap equation**

$$Z(\beta) = \sum_{\Delta} \rho(\Delta) e^{-\beta\left(\Delta - \frac{c}{12}\right)} = e^{\frac{\pi^2 c}{3\beta}} \tag{2.6}$$

   for $\rho(\Delta)$. The singular term on the RHS can teach us about asymptotics on the LHS: that's how the bootstrap works.

Outline

The Classical Bulk Solutions

Semiclassical AdS_3 gravity $\leftrightarrow$ CFT_2 with $c = \frac{3}{2G} \gg 1$. The spectrum is given by $\rho(E) \sim e^{S(E)}$, so we focus on $S(E)$ at leading order in c.

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Classical solutions to pure 3D gravity must be locally isometric to AdS_3:

1. **Global** AdS_3 corresponds to the vacuum (i.e. $\mathbb{1}$): $E = -\frac{1}{8G} = -\frac{c}{12}$.
2. **BTZ black holes** correspond to excited states (i.e. heavy primaries) and are formed from AdS_3 by discrete identifications: $E = M \geq 0$.

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   These have entropy in agreement with the Cardy formula **(?!?!)**:

$$S(E) = \frac{A}{4G} = 2\pi\sqrt{\frac{c}{3}E} \qquad (A \sim \sqrt{M}) \tag{3.1}$$

   Unlike the Cardy formula, this applies for *all* E, as long as $c \gg 1$.

The Classical Bulk Solutions

3. More generally, there are $\mathrm{SL}(2,\mathbb{Z})$ black holes that replace β by the complex modulus $\tau \in \mathcal{F} = \mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z}) \subset \mathbb{C}$.
4. One can apply large diffeomorphisms to excite "boundary gravitons" and create moving BHs. (In CFT, these correspond to descendants.)
5. Turning on matter, one finds conical defects with $-\frac{c}{12} < E < 0$, subleading horizonless geometries, and exotic black objects.

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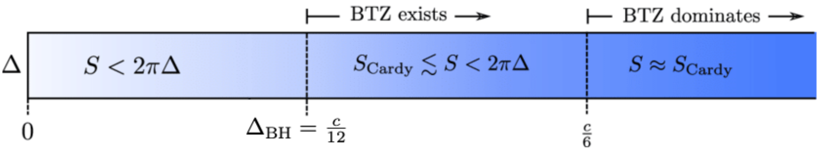

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Sparseness: Every holographic 2D CFT must have a "small" number of (primary) operators in the gap $-\frac{c}{12} < E < 0 \iff 0 < \Delta < \frac{c}{12}$.

In fact, we require only that $\rho(\Delta) \leq e^{2\pi\Delta}$ for $0 \leq \Delta \leq \frac{c}{12}$.

E.g. Consider $N = c \gg 1$ free bosons. We have $\rho(\Delta) = \exp\left(2\pi\sqrt{\frac{c}{3}\Delta}\right)$ for all $\Delta \geq 0$. There's no gap, so this theory cannot be holographic.

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The **Hellerman bound** states that every unitary, modular invariant 2D CFT must contain a primary with dimension $0 < \Delta_1 \leq \frac{c}{6} + 0.473695$. The bound has since been improved, and is expected to be $\Delta_1 < \frac{c}{12}$.

The Hawking–Page Transition

The spectrum in hand, we compute the partition function of 3D gravity:

$$Z_{\mathrm{AdS}}(\beta) = \sum_E \rho(E) e^{-\beta E} = e^{\frac{\beta c}{12}} + \int_0^\infty \mathrm{d}E\, \rho(E) \exp\left(2\pi\sqrt{\frac{c}{3}E} - \beta E\right).$$

This integral has a saddle point at $E_* = \frac{\pi^2 c}{3\beta^2}$, so to leading order in c,

$$Z_{\mathrm{AdS}}(\beta) = e^{\frac{\beta c}{12}} + e^{\frac{\pi^2 c}{3\beta}} \approx \max\left\{e^{\frac{\beta c}{12}}, e^{\frac{\pi^2 c}{3\beta}}\right\}. \tag{3.2}$$

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The thermodynamics is then exactly the same as before:

$$\beta F(\beta) = \log Z(\beta) = \frac{c}{12}\begin{cases} \frac{4\pi^2}{\beta}, & \beta < 2\pi \quad (\mathrm{BTZ}), \\ \beta, & \beta > 2\pi \quad (\mathrm{tAdS}). \end{cases} \tag{3.3}$$

The **Hawking–Page transition** occurs at $\beta_* = 2\pi$. At low T, the vacuum ("thermal AdS") dominates; at high T, BTZ is dominant. Of course, "small" BTZs still exist for $\beta > 2\pi$, where $E_* < \frac{c}{12}$.

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Gravity vs. CFT

Here's we know about $\mathrm{AdS}_3/\mathrm{CFT}_2$ so far:

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Theorem $(\mathrm{AdS}_3 = \mathrm{CFT}_2)$

We have $Z_{\mathrm{AdS}} = Z_{\mathrm{CFT}}$ iff 3 conditions are satisfied:

1. *The CFT is unitary and modular invariant;*
2. $c \gg 1$, *i.e. the limit $c \longrightarrow \infty$ is taken; and*
3. *The light spectrum is sparse, meaning $\rho(\Delta) \leq e^{2\pi\Delta}$ for $\Delta < \frac{c}{12}$.*

Proof of AdS/CFT

Proof

On the gravity side, we classified all bulk solutions as quotients of AdS_3. We determined their energies (there's a gap!), computed the entropy and density of states, and evaluated the partition function at large c:

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In the CFT, the key idea is to split $Z(\beta)$ into a sum of its contributions from light and heavy states, and then to use **modular invariance**:

$$Z_{\mathrm{CFT}}(\beta) = Z_{\mathrm{L}}(\beta) + Z_{\mathrm{H}}(\beta) = Z_{\mathrm{L}}\left(\frac{4\pi^2}{\beta}\right) + Z_{\mathrm{H}}\left(\frac{4\pi^2}{\beta}\right). \tag{4.2}$$

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Now at **large c**, the theory's degrees of freedom decouple. That is,

$$Z_{\mathrm{L}}(\beta) = Z_{\mathrm{H}}\left(\frac{4\pi^2}{\beta}\right), \qquad Z_{\mathrm{H}}(\beta) = Z_{\mathrm{L}}\left(\frac{4\pi^2}{\beta}\right). \tag{4.3}$$

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If $\beta > 2\pi$, then the last factor exponentially suppresses all terms except for $\Delta = 0$. Thus in fact Z_{L} is dominated by the vacuum, and we have

$$Z_{\text{L}}(\beta) = e^{\frac{\beta c}{12}} \implies Z_{\text{CFT}}(\beta) = e^{\frac{\beta c}{12}} + e^{\frac{\pi^2 c}{3\beta}} = Z_{\text{AdS}}(\beta). \tag{4.5}$$

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So $\text{CFT} = \sum_{\text{states}} (\text{fixed channel})$**, while** $\text{gravity} = \sum_{\text{channels}} (\text{vacuum})$**.**

Summary and Conclusions

- Modular invariance in 2D CFT is expressed by $Z(\beta) = Z(4\pi^2/\beta)$. The IR, where the vacuum lives, strongly constrains the UV.
- At high temperature, $Z(\beta)$ is dominated by the vacuum, so the thermodynamics is universal: $S(E) = 2\pi\sqrt{\frac{c}{3}E}$ (Cardy).

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- The classical solutions of pure 3D gravity are AdS_3 $(E = -\frac{c}{12})$ and the BTZ black holes $(E \geq 0)$. In particular, the spectrum is gapped.
- At large c, the partition function is $Z(\beta) = e^{\frac{\beta c}{12}} + e^{\frac{\pi^2 c}{3\beta}}$, and there is a Hawking–Page transition between tAdS and BTZ at $\beta_* = 2\pi$.

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- At large c, the partition function is $Z(\beta) = e^{\frac{\beta c}{12}} + e^{\frac{\pi^2 c}{3\beta}}$, and there is a Hawking–Page transition between tAdS and BTZ at $\beta_* = 2\pi$.
- Any holographic CFT must be unitary and modular invariant, have large central charge, and satisfy $\rho(\Delta) \leq e^{2\pi\Delta}$ for $\Delta \in [0, \frac{c}{12}]$.
- Under these conditions, the validity of the Cardy formula extends to all β, and moreover we have $Z_{\mathrm{AdS}}(\beta) = Z_{\mathrm{CFT}}(\beta)$.

Thank you for listening! Any questions?